Jefimenko Equations in Computation of Electromagnetic Fields for Lightning Discharges

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Abstract

The practicality of the Jefimenko equations in computations of electric fields, from lightning, was analyzed. The general Jefimenko equations are derived from Maxwell's equation. Different processes for the computations of electromagnetic fields due to lightning was compared with the generalized three dimensional Jefimenko equations. The applications of these generalized equations in the field of electricity and magnetism were assessed.

Keywords: 3 –D Jefimenko equations, Computation of electric fields, Maxwell's equations.

1. Introduction

A sudden electrostatic discharge that occurs between electrically charged regions between two clouds, cloud and air, and between a cloud and a ground is generally referred as lightning. In order for an electric discharge to occur two conditions are of utmost importance, they are a sufficiently high electric potential between two regions of space and a high resistance medium must obstruct the equalization of the opposite charges. As the thundercloud moves over the surface of the earth (which is considered as an infinitely conducting grounded plane) an equal electric charge of opposite polarity is induced on the earth surface. The oppositely charged create an electric field within the air between them. The greater the accumulation of charges, higher will be the electric field. The electric field created due to lightning is a complex process to comprehend. In general, the overall length of the lightning discharge is considered as a very thin channel. The underlying feature of lightning is that the fields are due to the moving charges; hence the magnetic and electric fields calculations require careful considerations of the retarded potentials. The lightning return stroke channel extends typically at a speed of about one third of the speed of light. Due to the finite time of the electromagnetic waves between the source and observer at any given time the remote observer sees the current of the return stroke channel form at an earlier time. Hence an effective method of determining the electric fields due to lightning is by the use of the concept of retarded potential. The uses of Jefimenko equations which are the result of retarded potentials in computations of electric fields due to lightning discharge have gained popularity at recent times. The goal of this paper is to analyze various changes that are bought about in the Jefimenko equations to make it favorable for the computations of lightning electric fields.

2. Jefimenko equations

Jefimenko equations, in the field of electromagnetism, describe the behavior of the electric and magnetic fields in terms of the charge and current distribution at retarded times. According to [6], Jefimenko equations are pair of equations that relate the electromagnetic fields in an observer's time frame to the charge and current sources in the retarded time frame. Jefimenko equations are the solutions of the Maxwell's equations for given electric charges and currents with the assumptions that only the charges and currents produce electric fields. The electric (***E***) and magnetic (***B***) fields from Jefimenko equations due to the arbitrary charge or current distribution of charge density (ρ) and the current density ($\boldsymbol{J}$) is given as follows.

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int \left[\left(\frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2 c}\frac{\partial\rho(\boldsymbol{r}',t_r)}{\partial t}\right)(\boldsymbol{r}-\boldsymbol{r}') - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|c^2}\frac{\partial \boldsymbol{J}(\boldsymbol{r}',t_r)}{\partial t}\right] d^3\boldsymbol{r}' \quad (1)$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_\circ}{4\pi}\int \left[\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|^2 c}\frac{\partial \boldsymbol{J}(\boldsymbol{r}',t_r)}{\partial t}\right] \times (\boldsymbol{r}-\boldsymbol{r}')d^3\boldsymbol{r}' \quad (2)$$

Where ***r'*** is a point in the charge distribution, ***r*** is a point in space, and

$t_r = t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}$, is the retarded time.

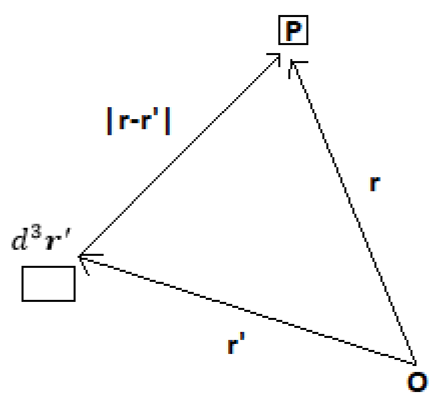

Figure 1: displacement for retarded potential.

These equations are time dependent generalizations of coulombs law and biot-savart law to electrodynamics. While time dependent potentials are simply the retarded forms of static potentials the time dependent fields are more than the retarded forms of coulombs and biot-savart law [3]. Maxwell's equations are generally understood as the bridge between the magnetic and electric fields, thus giving rise to a propagating electromagnetic wave for the simple reason that varying electric and magnetic fields can cause each other to change in time. However, Jefimenko equations show an alternating point of view in solving these time dependent electric and magnetic fields. Since from equations (1) and (2) the expressions for ***E*** doesn't depend upon ***B*** and vice versa making ***E*** and ***B*** fields impossible to create each other. Hence, there is no casual links between the electric and magnetic fields, instead describes that electromagnetic field is such a property that consists of electric and magnetic component simultaneously due to the common time varying electric charges and currents. Jefimenko equations not only hold for the time varying charges and currents but also when the charges and current distribution are static. Over recent period of time Jefimenko equations has earned a lot of popularity because of its applications other than lightning discharges. Some of these include electromagnetic calculation of nuclear explosions. Also,[1] has implemented the radiation component of the Jefimenko equations to compute the far field radiation on the ground for a high altitude nuclear explosion.

2.1 Retarded potential and Jefimenko equations

In electrodynamics, the retarded potentials are those electromagnetic potentials for the electric field generated by time-varying electric current or charge distribution in the past. As the electromagnetic information travels at the speed of light an observer at a distant from the charge distribution gets the "information" of earlier times and earlier position. In non-static case therefore, it's not the status of the source at present that matters, but rather, its position at some earlier timet_r, known as the retarded time, after the "message" left. Since the electromagnetic waves travel at the speed of light, the delay in travelling any distance,$|\boldsymbol{r}-\boldsymbol{r}'|$ is $\frac{|r-r'|}{c}$ hence, $t_r = t - \frac{|r-r'|}{c}$.

For time dependent fields, the retarded potentials as described by [2] is as follows,

$$\boldsymbol{V}(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int \frac{\rho(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau' \quad (3)$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_\circ}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau' \tag{4}$$

Where$\rho(\boldsymbol{r}',t_r)$ is the charge density that prevailed at point ***r'*** at the retarded time t_r and τ is less than or equals to time t. Because the integrals are evaluated at retarded time they are called retarded potentials[2]. From $V(\boldsymbol{r},t)$ and $\boldsymbol{A}(\boldsymbol{r},t)$ the fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ can be calculated using the definitions of potentials which eventually leads to Jefimenko equations as described in the proof of Jefimenko equations in chapter 3 of this paper.

3. Jefimenko equations from Maxwell's equation

Maxwell's equations are a set of partial differential equations that describe how electric and magnetic fields are generated by charges and currents. The major consequence of these equations is that they visualize how varying electric and magnetic fields propagate at the speed of light. These equations are named after the physicists and mathematician James Clerk Maxwell. He published an early form of these equations and first purposed the electromagnetic phenomenon of light. Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism.

The sources ρ, volume charge density and $\boldsymbol{J}$, volume current density generate the electric and magnetic field. Let $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are the electric and magnetic field intensity, $\mu_\circ$ and $\varepsilon_\circ$ are the permeability and permittivity of the medium. We know the Maxwell's equations are:

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t) = \frac{\rho(\boldsymbol{r}',t_r)}{\varepsilon_\circ} \tag{5}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0 \tag{6}$$

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \tag{7}$$

$$\nabla\times\boldsymbol{B}(\boldsymbol{r},t) = \mu_\circ\boldsymbol{J}(\boldsymbol{r}',t_r) + \frac{1}{c^2}\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t} \tag{8}$$

Where c is the speed of light which relates $c^2 = 1/(\mu_0\varepsilon_0)$

For scalar potential $V(\boldsymbol{r}, t)$ and vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ we can write

$$\boldsymbol{E}(\boldsymbol{r},t) = -\nabla V(\boldsymbol{r},t) \text{ and } \boldsymbol{B}(\boldsymbol{r},t) = \nabla \times \boldsymbol{A}(\boldsymbol{r},t)$$

Substituting these into Faraday's law, we get,

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = 0$$

Here as the above term whose curl gives zero, therefore it can be written as the gradient of scalar so,

$$\nabla^2 V(\boldsymbol{r},t) + \frac{\partial \nabla \cdot \boldsymbol{A}(\boldsymbol{r},t)}{\partial t} = -\frac{\rho(\boldsymbol{r}',t_r)}{\varepsilon_\circ} \tag{9}$$

where, t_r is retarded time. Again from the Maxwell's forth relation,

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) - \mu_\circ\varepsilon_\circ \frac{\partial^2 \boldsymbol{A}(\boldsymbol{r},t)}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) + \mu_\circ\varepsilon_\circ \frac{\partial V(\boldsymbol{r},t)}{\partial t}\right) = -\mu_\circ \boldsymbol{J}(\boldsymbol{r}',t_r) \tag{10}$$

These two equations (9) and (10) contain all the information in Maxwell's equation.

For the non-static cases,

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_\circ}\int \frac{\rho(\boldsymbol{r}',t_r)}{r} d\tau \text{ and } \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_\circ}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{r} d\tau$$

Where $\rho(\boldsymbol{r}',t)$is the charge density that prevailed at point ***r'*** at the retarded time where

$$t_r = t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c}$$

We know from the Lorentz condition,

$$\nabla \cdot \boldsymbol{A}(\boldsymbol{r},t) + \mu_\circ\varepsilon_\circ \frac{\partial V(\boldsymbol{r},t)}{\partial t} = 0$$

For any scalar function addition or subtraction is possible, due to this, there is no effect on the electric and magnetic field *(E* and *B*). Such changes in scalar potentials are called gauge transformation [2]. Hence the gradient of scalar potential gives,

$$\nabla V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_\circ}\int\left[-\frac{\dot{\rho}(\boldsymbol{r}',t_r)}{\mathrm{c}}\frac{\hat{\boldsymbol{r}}}{r}-\rho(\boldsymbol{r}',t_r)\frac{\hat{\boldsymbol{r}}}{r^2}\right]d\tau$$

By substituting $\nabla\rho(\boldsymbol{r}',t_r)=\dot{\rho}(\mathbf{r}',t_r)\nabla t_r=-\frac{1}{\mathrm{c}}\dot{\rho}(\boldsymbol{r}',t_r)\nabla r, \nabla r=\hat{\boldsymbol{r}}$ and

$$\nabla\frac{1}{r}=-\frac{\hat{\boldsymbol{r}}}{r^2}, \qquad \hat{\boldsymbol{r}}=\frac{\boldsymbol{r}}{r}$$

On taking the divergence of the gradient of scalar potential we know,

$$\nabla^2 V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_\circ}\int\left[\frac{1}{\mathrm{c}^2}\frac{\ddot{\rho}(\boldsymbol{r}',t_r)}{r}-4\pi\rho(\boldsymbol{r}',t_r)\delta^3(\hat{\boldsymbol{r}})\right]d\tau$$

$$\nabla^2 V(\boldsymbol{r},t)=\frac{1}{c^2}\frac{\partial^2 V(\boldsymbol{r},t)}{\partial t^2}-\frac{1}{\varepsilon_\circ}\rho(\boldsymbol{r}',t_r)$$

By substituting $\nabla\dot{\rho}(\mathbf{r}',t_r)=-\frac{1}{\mathrm{c}}\ddot{\rho}(\boldsymbol{r}',t_r)\nabla r$, $\nabla\cdot\frac{\hat{\boldsymbol{r}}}{r}=\frac{1}{\mathrm{r}^2}\nabla\cdot\frac{\mathrm{r}}{\mathrm{r}^2}=4\pi\delta^3(\hat{\boldsymbol{r}})$

where $\delta^3(\hat{\boldsymbol{r}})$ is a three dimensional $Dirac-delta\ function.$

The time derivative of vector potential $\boldsymbol{A}$ is

$$\frac{\partial\boldsymbol{A}(\boldsymbol{r},t)}{\partial t}=\frac{\mu_\circ}{4\pi}\int\frac{\boldsymbol{j}(\boldsymbol{r}',t_r)}{r}d\tau$$

Hence,

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_\circ}\int\left[\frac{\rho(\boldsymbol{r}',t)}{r^2}\hat{\boldsymbol{r}}+\frac{\dot{\rho}(\boldsymbol{r}',t)}{\mathrm{c}r}\hat{\boldsymbol{r}}-\frac{\boldsymbol{j}(\boldsymbol{r}',t_r)}{rc^2}\right]d\tau \qquad (11)$$

Equation (11) is the time dependent generalization of coulombs law.

For magnetic field,

$$\nabla\times\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_\circ}{4\pi}\int\left[\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{r^2}+\frac{\boldsymbol{j}(\boldsymbol{r}',t_r)}{rc}\right]\times\hat{\boldsymbol{r}}d\tau$$

Thus,

$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{\mu_\circ}{4\pi}\int\left[\frac{\boldsymbol{J}(\boldsymbol{r}',t_r)}{r^2}+\frac{\mathbf{j}(\boldsymbol{r}',t_r)}{rc}\right]\times\hat{\boldsymbol{r}}d\tau \qquad (12)$$

This is the time dependent generalization of Biot-Savart law. Hence the equations (11) and (12) are known as the Jefimenko equations derived from Maxwell's equations.

4. Electromagnetic fields of lightning discharges from Jefimenko equations

A general method of calculation of the time dependent electromagnetic fields was given by Lorentz in[4] in which the retarded potentials were first introduced. These are

$$\emptyset(\boldsymbol{x},t) = \int \frac{\rho(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|} d\boldsymbol{x}' \tag{13}$$

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{\mu_\circ}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{x}',t_r)}{|\boldsymbol{x}-\boldsymbol{x}'|} d\boldsymbol{x}' \tag{14}$$

Where $\emptyset$ and$\boldsymbol{A}$are the scalar and vector potentials in Gaussian units,ρ and$\boldsymbol{J}$ are the charge and current densities. According to[5], Lorentz didn't explicitly display the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$, though he noted they could be obtained from equations, $\boldsymbol{E} = -\boldsymbol{\nabla}\emptyset - \frac{1}{C}\frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. [5] further explained that had Lorentz work been better received by Maxwell, the time dependent electric and magnetic fields have been known a century ago, but Maxwell couldn't understand the prospect that time-dependent potentials were useful tools and hence deserved models of their own.

The seminal paper [8] in the computation of electromagnetic equations by lightning, unfolded new possibilities in the field of lightning discharges. Paper [8] derived a pair of time dependent electromagnetic equations for a one dimensional current element given by

$$d\boldsymbol{E}(\boldsymbol{r},t) = \frac{dz'}{4\pi\varepsilon_\circ}\left\{cos\theta\left[\frac{2}{r^3}\int_0^t I\left(z',\tau-\frac{r}{c}\right)d\tau + \frac{2}{cr^2} I\left(z',\tau-\frac{r}{c}\right)\right]\boldsymbol{a_r} + sin\theta\left[\frac{1}{r^3}\int_0^t I\left(z',\tau-\frac{r}{c}\right)d\tau + \frac{1}{cr^2} I\left(z',\tau-\frac{r}{c}\right) + \frac{1}{c^2 r}\frac{\partial I\left(z',\tau-\frac{r}{c}\right)}{\partial t}\right]\boldsymbol{a_\theta}\right\} \tag{15}$$

$$d\boldsymbol{B}(\boldsymbol{r},t) = \frac{\mu_\circ dz'}{4\pi} sin\theta\left[\frac{1}{r^2} I(z',\tau-r/c) + \frac{1}{cr}\frac{\partial I(z',\tau-r/c)}{\partial t}\right]\boldsymbol{a_\varphi} \tag{16}$$

Where dz' is the 1-D line element of current, $\boldsymbol{r}$ and r are the vector and distance from the distance dz' to the observer, t is the time in the observer's frame of reference, θ is the angle measured from dz' to $\boldsymbol{R}$, I is the current magnitude along z' and $\boldsymbol{a_R}$, $\boldsymbol{a_\theta}$ and $\boldsymbol{a_\varphi}$ are the unit vectors in R, θ *and* φ directions in a spherical coordinate frame, respectively.

Thottappillil in [7] have described that, "These equations are fundamentals in lightning electromagnetism and are widely and successfully used to investigate the spatial and temporal evolution of the lightning discharge current from remote electromagnetic measurement". However, these equations were derived for an infinitely thin, one dimensional source current and not for a general three dimensional current distributions. Hence, for the generalization of these lightning electromagnetic equations paper [6] introduced a corresponding pair of generalized equations that are determined from the three dimensional time dependent current density distribution based on Jefimenko original electric and magnetic field equations. For this, he made some adjustments so that the Jefimenko electric field equation depends only on the time dependent current density rather than on both the charge and current densities in its original form. According to[6], the generalized three dimensional electric field equation is represented as

$$\boldsymbol{E}(\boldsymbol{x},t) = \frac{1}{4\pi\varepsilon_\circ}\int_0^t d^3x' \left\{ \int_0^t \frac{2\left(\boldsymbol{J}\left(\boldsymbol{x}',\tau-\frac{r}{c}\right)\cdot\hat{\boldsymbol{r}}\right)\hat{\boldsymbol{r}} + \left(\left(\boldsymbol{J}\left(\boldsymbol{x}',\tau-\frac{r}{c}\right)\times\hat{\boldsymbol{r}}\right)\times\hat{\boldsymbol{r}}\right)}{r^3} d\tau \right.$$

$$+ \frac{2\left(\boldsymbol{J}\left(\boldsymbol{x}',t-\frac{r}{c}\right)\cdot\hat{\boldsymbol{r}}\right)\hat{\boldsymbol{r}} + \left(\boldsymbol{J}\left(\boldsymbol{x}',t-\frac{r}{c}\right)\times\hat{\boldsymbol{r}}\right)\times\hat{\boldsymbol{r}}}{cr^2}$$

$$\left. + \frac{\left(\frac{\partial \boldsymbol{J}\left(x',t-\frac{r}{c}\right)}{\partial t}\times\hat{\boldsymbol{r}}\right)\times\hat{\boldsymbol{r}}}{c^2 r} \right\} \quad (17)$$

Hence, equation (17) and the original Jefimenko magnetic field equations (12) stated in the previous topic compose a new pair of Jefimenko equations and can be considered as the generalized versions of the equations described in [8]. The first term of equation (15) with the time integration is similar to that of the time dependent static field (quasi-static) of electric dipoles. The second term with the $1/r^2$ dependence is the same as the induction field of electric dipoles and the third term with $1/r$ dependence represents the radiation field. Similarly, the derivation of equations described in [8] from the generalized Jefimenko equations described in [6] further solidifies the flexibility of Jefimenko equations in computations of electromagnetic fields of lightning discharges.

5. Discussion and Conclusion

Thus, the generalized 3-D form of the Jefimenko equations described in [6] clarified a lot of confusion discussed in [2], [5]and the reference therein including the assumptions that were considered in solving the electric fields due to lightning by[8].These generalized equations were represented in volumetric current distribution rather than a thin, channeled current [6] represented the lightning discharge in a 3D conical shape with open angle of the cone assumed to be 60° and its axis assumed to point downward. This made visualization and computation of lightning less cumbersome. The new generalized Jefimenko equations have been solely expressed in the form of current density with a simple reason that current densities can be relatively easily measured, but charge densities cannot be easily measured. The clarification of Jefimenko electric field equation in terms of time-dependent current density is done by [5] using the continuity equation.[6] described that in the case of thin channels, even though the equations (15) and (17) are mathematically equivalent, he found it is more straight forward and efficient to code the calculation of vector form (17) than the angular form (15). We can also introduce a true static term in equation (17) which represents a static electric monopole field similar to that of the Coulombs law. It is therefore clear that the practical use of the generalized 3-D Jefimenko equations is immense at present as these equations have been applied to different sectors of electromagnetism including the calculations of lightning discharges, electromagnetic calculations of nuclear explosions etc. Thus, Jefimenko equations describes a clear and concise expressions for the electric and magnetic fields that is clearly defined in terms of the conventional concept of static, induction, and radiation fields.

6. References

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